THE MATHEMATICS TEACHER

The Official Journal of
The National Council of Teachers of Mathematics
Incorporated 1928

JANUARY 1934

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Volume XXVII **Number 1**

Published by the
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
Menasha, Wisconsin New York

Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412 P. L. & R., authorized March 1, 1930.

THE MATHEMATICS TEACHER is published monthly except June, July, August and September. The subscription price is $2.00 per year. Single copies sell at 40 cents each.

J. Dailly Esqr. Sculp.

BROOK TAYLOR, L.L.D. & R.S.S. 1714.

London, Published March 26, 1811, by J. Taylor, High Holborn.

THE MATHEMATICS TEACHER

Volume XXVII Number 1

Edited by William David Reeve

Changes in the Teaching of Geometry and Why We Teach It

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EUCLID'S METHODS COMPARED WITH THOSE OF THE PRESENT DAY

Definitions

EUCLID PREFACED EACH BOOK with a set of definitions. The present preference from the pedagogic standpoint is to introduce them as needed, as Lamy [1685] did in the "Elemens of Lamy."[1]

Number of Assumptions

Much controversy has arisen concerning Euclid's assumptions, especially about the fifth postulate. The modern trend in the secondary school is toward freedom in the use of assumptions and, in general, an increase in their number.

Hypothetical Constructions

Euclid does not admit hypothetical constructions, i.e., the use of constructions which have not first been shown possible and done beforehand in the geometric sequence adopted. Modern writers do. In the latest book on geometry,[2] practical constructions used by

[1] F. W. Kokomoor, "Teaching of Elementary Geometry in the Seventeenth Century," Isis, II (1928), pp. 94, 99.

[2] Smith, Reeve, and Morss, "Text and Tests in Plane Geometry," New York: Ginn and Co., 1933, I+286 pp.

draftsmen are given for the drawing of figures. Afterwards, when the student has acquired an idea of proof, proofs are given for the corresponding geometric constructions.

Ruler, Compasses and Mechanical Devices

Plato "objected to the use of mechanical devices in geometry and, according to tradition, he restricted the subject to the study of those figures which could be drawn by means of compasses and an unmarked straightedge."[3] Euclid did likewise. This excluded the spirals of Archimedes and the conics of Apollonius, treated later by these men. The restriction still holds theoretically but, as just stated, practical everyday methods are now given for "drawing a figure" to be studied and in these any desired instruments may be used. For "constructing a figure" to meet given geometric requirements, that is in a problem in geometry, only the straightedge and cor passes are allowed and proof must be given to show that the requirements are met. A distinction is thus made between "drawing a figure" and "constructing a figure."[4] In the latter, the ruler and compasses restriction still holds. The use of devices involving geometric principles[5] and the use of models[6] are both considered good pedagogy.

Inclusion of Applications

Euclid's work is "pure"[7] subject matter. He does not describe applications of geometry: "astronomy, geography, architecture, machinery, optics and military tactics"[8] together with illustrations of various geometric forms common in life and nature.[9] However, such applications and illustrations are a part of the present method, the object being to link geometry and other branches not only with the future but with the past of the learner's life experience.

[3] Vera Sanford, "A Short History of Mathematics," p. 9.

[4] Smith, Reeve, and Morss, "Text and Tests in Plane Geometry," p. 18.

[5] F. W. Kokomoor, op. cit., p. 100.

[6] Georg Wolff, "The Mathematical Collection," *Eighth Yearbook of the National Council of Teachers of Mathematics*, pp. 216–243.

[7] John A. Swenson used the word "pure" in this sense in his course: "Professionalized subject matter in senior high school mathematics," Teachers College, Columbia University, 1933.

[8] F. W. Kokomoor, op. cit., p. 100.

[9] Done in Smith, Reeve, and Morss, "Text and Tests in Plane Geometry."

Proofs by Geometry and by Algebra

Excessive use of symbols can complicate the teaching process but the introduction and use of symbols of algebra and arithmetic has rendered present day treatment of some of the material in the "Elements" a shorter and simpler matter than the geometric method of demonstration used by Euclid. The older method, however, gives concrete illustration to matters that, treated algebraically or arithmetically, could be considered abstract and is used to-day in algebra under the name of graphic algebra.[10]

Generalization

According to Sir T. Percy Nunn, there are two kinds of generalization: one not founded upon analysis and the other which is. Analysis separates out the "essential process"[11] of a particular case. Let us distinguish between these two types:

1. *The generalization not founded upon analysis* is *not* dependable. The probability of its truth increases with the number of cases to which it applies which agree with the results given by the generalization. If one case to which it applies is found which does not agree with the results obtained from the generalization, the latter collapses. This generalization was not founded upon a recognition of the "essential process" of a particular case, i.e., not built on analysis.

2. *The generalization founded upon analysis* recognizes that the "essential process" of the particular case applies to *all* problems of the same type. Only a single case is needed to make it certain since it is built on an analysis of the circumstances on which the relation depends. This dependable kind of generalization is the type that modern methods aim to use.

One of De Morgan's criticisms of Euclid was that he failed to employ generalizing notions.[12] As an instance, he defines an angle as the sharp corner between two lines that meet and thus does not include the straight angle and the reflex angle. Our age is an age of generalization or of the concentration of knowledge into small parcels. The formulas of algebra and trigonometry, the very alge-

[10] Sir T. Percy Nunn, "The Teaching of Algebra" (including Trigonometry), page 70.

[11] T. Percy Nunn, op. cit., p. 2.

[12] Alva Walker Stamper, "A History of the Teaching of Elementary Geometry with Reference to Present Day Problems," p. 30.

braic symbols themselves, "thought of as substitutes, not immediately for numbers, but for words as the vehicles of general ideas."[13] and generalizations of geometric theorems are all examples in point. We say in the small shorthand sentence called a formula the essence of volumes. We concentrate *all* the theorems concerning the three sides and an angle of a triangle including the cases in which the angle under consideration is 0°, acute, right, obtuse and 180° into *one* theorem which we call the *Law of Cosines*. The Law of Cosines may be stated as follows:

> *The square on the side opposite an angle of a triangle is equal to the sum of the squares on the two sides including that angle diminished by twice the product of those two sides and the cosine of their included angle.*

This law, in turn, we may concentrate into still shorter form by means of the symbols of algebra in which a, b, c represent the sides of the triangle and A, B, C represent respectively the angles opposite these sides. Thus, in algebraic symbols, we write the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Since the Law of Cosines is true for any angle of the triangle, we can write in the same way:

$$b^2 = c^2 + a^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

To those who understand the symbols, however, the first equation or any one of the others, brings to mind, not only the one case which involves the particular sides and angle mentioned in that particular equation but three cases in which the three angles in turn appear. It is for this reason that we may say, in the language of symbols, that the Law of Cosines is: $a^2 = b^2 + c^2 - 2bc \cos A$. In the case in which $A = 90°$, we have the theorem of Pythagoras, the last term on the right disappearing, since $\cos 90° = 0$.

Analysis, Synthesis and Thinking

According to Christofferson,[14] "analysis is a mental process of tearing down a geometric statement to discover the relationships upon which its truth or existence depends."; while "synthesis is the building up of sufficient reasons to establish or prove the con-

[13] T. Percy Nunn, op. cit., p. 6.

[14] "Geometry Professionalized for Teachers," p. 55.

clusion." Plato is credited with originating and using the analytic method before the time of Euclid. Euclid used the synthetic method in the "Elements" but he, no doubt, used the analytic method in discovering some of the proofs that he later presented synthetically.

Geometric analysis supposes true the statement to be proven. We say: "If this is true, then this is true; but if this is true, then this is true" and so on until we arrive at something that we already know or can prove to be true or accept as true. It is as though we have broken the statement to be proven and supposed true into component parts which are connected by a chain of reasoning of the "If-Then" type.[15] In synthesis, we start with *this* something just arrived at in analysis that *we know to be true or have accepted as true* and by a logical chain of reasoning of a type that I shall call the Since-Then kind, piece together into what we call a synthetic proof the parts that we have found by analysis. The final statement in the synthetic proof is that which we started to prove,—in analysis, we supposed it true; in synthesis, we have proven it true. The parts into which we separated it by "If-Then" reasoning in analysis, we have pieced together by Since-Then reasoning in synthesis to form the proof of that which we started to demonstrate. Analysis, Synthesis and Generalization are dealt with, more fully than the title of this article will permit, in an article that I hope to send to press shortly.

The synthetic proof gave opportunity for the concise, logical presentation of the mathematics of his day that Euclid desired. Although possessing the advantage of brevity and coherence, it does not, however, give the method of attack through which it was deduced and therefore the analytic or exploration method which does this is helpful in the presentation of geometry to the learner. For a long time, the synthetic method has been the accepted method of presentation adopted by the writers of geometry texts. W. D. Reeve has suggested the presentation in texts of both analysis and synthesis as used in connection with problems in order that the student may learn to use both.[16] Stating the plan of proof before giving the synthetic demonstration is a step in this direction.[17]

[15] C. J. Keyser, "Thinking About Thinking," p. 18.

[16] In the major course: "The teaching and supervision of mathematics, Junior high school", Teachers College, Columbia University, Summer Session, 1932.

[17] This method is used in the Smith, Reeve, and Morss book, op. cit.

Euclid's Aim

As David Eugene Smith told us at the Mathematics Conference,[18] Euclid's aim was to teach logic so that it would be understood by the young men who came to the University in which he taught at Alexandria. To construct a set of logical propositions demands the acceptance of some unproved assumptions from which to make a start.

Is Euclid's Set of Assumptions "Scientific"?

According to J. Shibli,[19] "A perfectly scientific system of assumptions" should possess the following three fundamental properties:

1. *Consistency, meaning that no axiom contradicts the others, and that propositions deduced from the system lead to no contradictions.*
2. *Independence, meaning that no axiom could be deduced from the others.*
3. *Completeness, meaning that all the axioms used in the science are explicitly stated so that all the theorems can be derived from them without further appeal to intuition.*

Euclid's system, according to Shibli, meets the tests of consistency and independence but is imperfect with regard to completeness.

The Modern Aim

The first aim of modern methods is to teach in accordance with what are considered, at the time, by the teacher, to be the best pedagogic principles. At present, the aim is to teach logic but not until the mind of the pupil is ready for logic. Euclid wrote for mature minds.

Are Modern Sets of Assumptions "Scientific"?

The modern sets of assumptions are flexible and may change with the age and ability of the pupil. The Committee of Fifteen recommended more postulates and the assumption or informal treatment of twelve of the propositions. In 1923, the National Committee on Mathematical Requirements gave a list of sixteen statements which may be assumed. In his frequently-quoted address before the British Association, thirty years ago, John Perry

[18] Held at Teachers College, Columbia University in the City of New York, July 20, 1933.

[19] "Recent Developments in the Teaching of Geometry," pp. 96–98.

advocated the increase of assumptions. Carson,[20] another British educator, advises the adoption of "easily acceptable" theorems as postulates, later reducing the number of such postulated theorems after speculation becomes natural and after the meaning of consistence and independence have been explained. The unanimous report of the Teaching Committee of the Mathematical Association of Great Britain advises an increase in the number of assumptions and the use of hypothetical constructions.[21] Euclid, it was noted earlier, did not admit hypothetical constructions in his geometric structure.

The British report advocates for students of sixteen or seventeen years, a "rounding off" course to systematize their knowledge of plane geometry; to arrange it in a logical sequence dependent on a small number of axioms. This is the stage, called "The Systematizing Stage" in the British Report.[22] It was for this stage that Euclid wrote the "Elements."

Ideal Material for Training in Logic and Demonstration

As we shall see,[23] Euclid's "Elements" bring in the geometric equivalent of algebra, theory of numbers and solid geometry with which the above references concerning recommendations dealt.

Since all topics handled in the "Elements" are dealt with geometrically, it is plain that the first Book of the "Elements" had therefore to deal with geometry and the simpler parts of it in order to provide the foundation required for the logical development of the work. For this reason, it dealt with points, lines, surfaces and the simpler figures arising through combinations of them. The figures being familiar to all from childhood are simple for beginners to visualize and they remain the same 'yesterday, to-day and forever.' [See note 35.]

The first Book of the "Elements" dealt with basic material essential for Euclid's sequence in the "Elements" but, in addition, the above facts that made it ideal material as the vehicle of Euclid's training in logic, make it also ideal material for instruction in

[20] "Essays on Mathematical Education," p. 11.

[21] Hypothetical constructions are explained at the first part of this article.

[22] "The Teaching of Geometry in Schools," A Report Prepared for the Mathematical Association, London, G. Bell and Sons, Ltd., p. 16.

[23] The author's article on "The Elements of Euclid," to be published in a later issue of the Mathematics Teacher.

demonstrative geometry, that is for training in the use of demonstration.

For this reason too, it is the best known part of the "Elements" —that part of which many think, when the "Elements" are mentioned. Since Euclid dealt with all the topics treated in the "Elements" from a geometric standpoint, we might say that the "Elements" are a comprehensive treatment of plane and solid geometry.

Why Euclid Did Not Call the "Elements" Geometry

It should, however, be remembered that:

Euclid did not call his treatise a geometry, probably because the term still related to land measure, but spoke of it merely as the "Elements."* Indeed, he did not employ the word "geometry" at all, although it was in common use among Greek writers.[24]

Sequence

The editions of Euclid and books on geometry of Euclid's successors, for many centuries, show an adherence to his sequence. According to G. St. L. Carson[25] and the Report of the National Committee on Mathematical Requirements, mathematicians are now agreed that geometry need follow no one definite sequence and present day writers are presenting demonstrative geometry in various sequences. The value of preceding demonstrative geometry by intuitive geometry is also being recognized and some writers include both in one text.[26]

Intuitive Geometry

It should be noted, at this point, that "Mathematics owes its great forward steps to the intuition of its giants."[27] Young puts it, as follows:[28] "For with all our insistence on the formal logical

* Smith tells us in note[2] that Plato and Herodotus used the word "geometry" "in some of its forms, but always to indicate surveying."

[24] Smith, David Eugene, "History of Mathematics," Vol. II, Special Topics of Elementary Mathematics, p. 273.

[25] "Essays on Mathematical Education." In 1903, the University of Cambridge in its enumeration of the geometrical knowledge required for entrance to the University said that: any proof which appeared to form part of a systematic treatment of the subject would be accepted.

[26] For example, Smith, Reeve, and Morss, "Text and Tests in Plane Geometry."

[27] Smith and Reeve, "The Teaching of Junior High School Mathematics," p. 137.

[28] J. W. Young, "Lectures on Fundamental Concepts of Algebra and Geometry," p. 221.

procedure, the important fact must not be lost sight of that formal logic is in only a small minority of cases the method of mathematical discovery. Imagination, geometric intuition, experimentation, analogies sometimes of the vaguest sort, and judicious guessing, these are the instruments continually employed in mathematical research." In view of these facts, the development of intuition is an important factor in education for the future growth of mathematics. Sooner or later, everyone has to measure things and since intuitive geometry provides training in measurement, we may say that: "Intuitive geometry is, therefore, a subject to be required of all because everyone uses it."[29]

The demonstrative geometries we now have are based upon those that were written for a pupil who was chronologically at least a year or two (and mentally older still) than the one who is now studying geometry in the United States.[30]

Demonstrative Geometry

Its Purposes

We have already seen that Euclid's aim in writing the "Elements" was to build the mathematics of his time into a logical whole around notions already familiar to those for whom he wrote.

The modern endeavor, in connection with demonstrative geometry, can best be shown by a few quotations from writers of our day.

W. D. Reeve says:[31]

We must continually repeat and emphasize the fact that the purpose of geometry is to make clear to the pupil *the meaning of demonstration, the meaning of mathematical precision, and the pleasure of discovering absolute truth.*

C. W. Upton tells us:[32]

. . . on the part of the more progressive teachers to-day, the purpose in teaching geometry is not only to acquaint pupils with the methods of proving geometric facts, but also to familiarize them with that rigorous kind of thinking that Professor Keyser has so aptly called 'the If-Then kind, [C. J. Keyser, "Thinking About Think-

[29] Smith and Reeve, "The Teaching of Junior High School Mathematics," p. 138.

[30] Ibid., p. 137. Written before the book mentioned in note 2 was published. See bibliography.

[31] "The Teaching of Geometry," The Fifth Yearbook of the National Council of Teachers of Mathematics, p. 13.

[32] "The Use of Indirect Proof in Geometry and in Life," The Fifth Yearbook of the National Council of Teachers of Mathematics, p. 131–132.

ing", p. 18. Present writer's note] a type of thinking which is distinguished from all others by its characteristic form: *If* this is so, *then* that is so'.

Professor Upton continues that Professor Keyser has clearly pointed out that:

Euclid's great contribution was not only to geometry but to *a method of thinking* which is applicable not only in mathematics, but also 'in every other field of thought—in the physical sciences, in the moral or social sciences, in all matters and situations where it is important for men and women to have logically organized bodies of doctrine to guide them and save them from floundering in the conduct of life.'

W. S. Schlauch, in discussing "Why We Teach Demonstrative Geometry" states that:[33]

Geometry seems, of all secondary school subjects, best adapted to initiate a student into the meaning of mathematics as the science of necessary conclusions. A training in logical processes and a knowledge of what is meant by a complete proof of a proposition, based on given data, are as necessary for the average citizen as a study of economics or civics. How can he draw valid conclusions in these fields unless he has had some training in logical processes? Geometry furnishes the best available material for this training, varying from the simplest to the most complex; starting with a few assumptions or axioms, and building a logical system which results in a body of established truths which can be used to establish further truths.

G. D. Birkhoff and R. Beatley put it: "in demonstrative geometry the emphasis is on reasoning."[34]

In retrospection, David Eugene Smith tells us of his own education:[35]

What I learned in chemistry, as a boy, seemed true at the time, but much of it today is known to be false. What I learned of molecular physics seems at the present time like children's stories, interesting but puerile. What we learn in history may be true in some degree, but is certain to be false in many particulars. So we may run the gamut of learning, and nowhere, save in mathematics, alone, do we find that which stands as a tangible symbol of the immortality of law, true 'yesterday, today, and forever.'

We may change the symbols, . . . they are temporary expedients to convey the idea; we may speak in different tongues, . . . they are local expedients to convey thought; but it is inconceivable to us that the relation which the formula expresses should not be true always and everywhere, . . . a tangible symbol of the immortality of law.

[33] "The Analytic Method in the Teaching of Geometry," The Fifth Yearbook, p. 134.

[34] "A New Approach to Elementary Geometry," The Fifth Yearbook of the National Council of Teachers of Mathematics, p. 86.

[35] "Religio Mathematica," Mathematics Teacher, Vol. XIV, Dec. 1921, pp. 416–417.

The National Committee on Mathematical Requirements states that:

> The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual.

H. R. Hamley[36] tells us that:

> The science of mathematics has to do with certain fundamental concepts of number and space, with logical propositions derived from these concepts and with practical applications of the propositions so derived to everyday computation and measurement.

In geometry, this definition calls for both intuitive geometry and demonstrative geometry, the former dealing with "everyday computation and measurement," the latter dealing with "fundamental concepts" and "logical propositions" derived from them.

The Present Day Demand and How that Demand Is Met

The present day demand is for interesting, challenging subject matter that will prepare both for life and for further mathematical study. To achieve this, algebra, geometry, trigonometry, calculus and other subjects are organized together in such a way as to facilitate the process of learning. The aim is to teach mathematics, not algebra, nor geometry, nor trigonometry nor any other *part* of mathematics, but mathematics which includes them all, organized in such a way as to increase the wonder motive and to finally give the student a grasp of mathematics as a whole. There seems to be a difference of opinion as to what the resultant material should be called—general mathematics, correlated mathematics, combination mathematics, fused mathematics, unified mathematics, cooperative mathematics and other names having been applied to it by writers and teachers. Perhaps, we shall finally call the result mathematics, since that is what it is and omit qualifying adjectives other than elementary, intermediate and advanced or junior high school and senior high school. Algebra and geometry as taught in the past, are parts of mathematics and not mathematics as a whole.

[36] "The Function Concept in School Mathematics, "Unpublished Ph.D. thesis of Teachers College, Columbia University, 1932.

The attitude of many teachers of to-day, is that given by John A. Swenson,[37] in discussing the "Difference Between Algebra and Geometry." Swenson says:

> Modern inventions have produced such powerful tools that it is hardly excusable to confine onself exclusively to those in use 2,000 years ago. If Euclid lived to-day he would no doubt take advantage of modern algebraic methods. When the modern teacher of mathematics keeps algebra and geometry in separate compartments, he sacrifices much that a union of the two subjects can bring about.

Euclid aimed at logical perfection, present day writers aim at pedagogical perfection combined with logic. Stamper has said that:[38] "A sequence that is logically best is not necessarily pedagogically best." The same idea is voiced by quotations from other writers given earlier in this paper. The attempt to realize pedagogical perfection and provide logical training has been responsible for the mathematics described in this section.

It should be remembered that geometry furnishes a training in logic that algebra fails to provide since the manipulation of symbols in algebra obscures the underlying logic from view. In this connection, a quotation from the article by W. D. Reeve, quoted earlier, answers clearly and definitely a question that sometimes arises. Reeve says:[39] "Someone may ask, If training in constructive thinking is the big objective, why not give a course in pure logic? The answer is that geometry furnishes appropriate figures to illustrate and apply the essential types of thinking, while pure logic does not." As mentioned earlier, geometry is, if approached psychologically, available for the childhood of the race, since it handles concepts arising from forms with which children are already familiar. Since then, geometry furnishes logical training in connection with definite, familiar material, it seems that geometry, in logical sequence, should, as often as possible, form a part of the child's school training. Text and Tests in Plane Geometry[40] furnishes illustrative material from widely distributed sources along with training in logic and generously meets the need for testing that is

[37] "Graphic Methods of Teaching Congruence in Geometry," Fifth Yearbook of the National Council of Teachers of Mathematics, pp. 100–101.

[38] "A History of the Teaching of Elementary Geometry, With Reference to Present Day Problems," p. 141.

[39] "The Teaching of Geometry," The Fifth Yearbook of The National Council of Teachers of Mathematics, p. 14.

[40] See note 2.

well expressed by E. R. Breslich as follows:[41] " . . . the subject matter should be subjected to a careful program of testing and reteaching to secure objective evidence of what has been learned."

Exercises

In the effort to meet the demands of pedagogy, exercises or originals provable by the use of results already proven in the geometric sequences adopted are now introduced into geometry texts and the teaching of geometry. Loney was among the first to utilize this method. Euclid with his aim at training in logic, had no such original exercises included in the "Elements," feeling, no doubt, that such might detract from the coherence of the logical whole that he was building and which, so far as he himself was concerned, was a succession which included a good many originals.

Individual Differences

The important problem of handling individual differences which rises as soon as we aim at pedagogic perfection, must have appeared to the conscientious teacher at the very dawn of teaching. It was present in the days of Pythagoras, and he handled the matter by forming "the brotherhood of the Pythagoreans"[42] which "seems to have been a secret society with an inner group of men admitted to full membership and an outer circle of people who knew only part of the mysteries."[42]

Euclid handled it by providing the sequence of the thirteen books of the "Elements" in which the students might go as far as their several abilities and inclinations dictated, thus automatically finding their own levels.

Some present day leaders[43] advocate having A, B, C subdivisions of a class in the one room. Teaching these at different times allows the B and C groups "to pick up some crumbs of learning" from the teaching of the A group and the C group to do likewise from the teaching of the A and B groups. All the groups know that a member of either the B group or the C group will be changed to a higher group as his work merits his promotion.

[41] "Articulation of Junior and Senior High School Mathematics," The Eighth Yearbook of The National Council of Teachers of Mathematics, p. 4.

[42] Vera Sanford, "A Short History of Mathematics," p. 7.

[43] Professor W. D. Reeve suggested this plan in the course: The teaching and supervision of mathematics in the senior high school, Teachers College, Columbia University, Summer Session, 1933.

In another method,[44] three levels of courses are provided, named set C, set B and set A, each in turn dealing more deeply with the same topic. Students start together on the same topic at set C and go through sets C, B, and A as their individual achievements warrant. Students thus place themselves in accordance with the quality and quantity level of the work done by them.

Others provide texts containing extra materials, both in subject matter and exercises, for the quicker students.

Some teachers provide two classes in the same subject matter. The first in which the whole class participates which meets the needs of the brighter students, is followed, at another time, by a second attended by those who did not understand all that went on in the first class and the students thus automatically separate themselves into groups.

Another method is to aim, in the regular classes, to meet the needs of the average group rather than of the brightest group. Then two additional classes are held—one for those who wish to go more deeply into the subject and one for those students who require additional teaching to enable them to thoroughly assimilate what was done in the regular class. We have, again, automatic separation decided by the student. The two additional classes are open to all the students of the class and often members of the A group attend both and sometimes members of the average group who aspire to higher things do also.

Others handle the problem by providing additional, educational projects which supplement the classwork and in which a student may or may not participate as he or she wishes.

Sometimes, a teacher uses several of these methods in connection with the teaching of one class; sometimes the methods are used separately, using different methods or the same method in the work with different classes as the calibre of the class determines. The teacher needs to know her tools, how to use them and then be able to utilize them as conditions demand.

The British Report on "The Teaching of Geometry in Schools"[45] divides the teaching of geometry into five stages: Stage A: The Experimental Stage, Stage B: The Deductive Stage, Stage C: The Systematising Stage, Stage D: Modern Geometry, Geometrical

[44] J. T. Johnson, "Adapting Instructional Material to Individual Differences in Learning," The Mathematics Teacher, Volume XXVI, Number 4, April, 1933.

[45] A Report prepared for the Mathematical Association.

Conics and Formal Solid Geometry, Stage E: The Philosophy of Geometry. The particular class in which a student is enrolled depends on which one of the five stages of geometrical growth has been attained by the student; this and his proceeding further, in turn, depending on the particular student's geometrical aptitude and inclinations. Stages D and E fall outside the domain of elementary mathematics,[46] as usually taught in American secondary schools.

The problem of handling individual differences will be present as long as teaching continues. The problem increases in magnitude as the per cent of the total population which forms the student class increases; since the students do not form so select a group as they did in earlier days when aptitude, a desire to learn and the possession of the determination to acquire an education were the factors which decided whether or not one went on to high school and to college.

Needed: Better Trained Teachers

W. D. Reeve has told[47] us on several occasions, that the need of mathematics to-day is better trained teachers. This is undoubtedly true. E. R. Breslich says:[48] "Each teacher must know what has preceded his course and build on it. Each must be familiar with that which is to follow and must pave the way for it." He also says, in discussing the teaching of one branch of mathematics, that of graphing,[49] "On the part of the teachers, it presupposes a thorough knowledge of the subject." In the same article, Breslich writes:[50] " . . . teachers are not qualified to teach junior high school mathematics unless they have a thorough knowledge of the mathematics taught in the senior high school and in the junior college. Without this knowledge they will lack a comprehensive point of view, and their work will be ineffective." The first two quotations given above from Breslich's article make it apparent that Breslich also presupposes on the part of the junior high school teacher, a thorough knowledge of the mathematics being taught by the teacher as well as the mathematics of the earlier grades. One might go on and say that to teach senior high school mathematics, one requires, at

[46] Ibid., p. 16.
[47] At Teachers College, Columbia University, New York City.
[48] Op. cit., p. 13.
[49] Ibid., p. 8.
[50] Ibid., p. 7.

least, a thorough knowledge of mathematics of the earlier grades, the junior high school, the senior high school and the four year college.

In the words of W. C. Bagley,[51] the teacher requires "a thoroughly firm grasp of his teaching materials."

It is the same in all subjects. John Thompson, in discussing "How to Make the Most of Class Piano Teaching"[52] says: "It is impossible to teach what one does not know oneself."

In dealing with "The Ideal Preparation of a Teacher of Secondary Mathematics from the Point of View of an Educationist," William C. Bagley concludes:[53]

> It goes without saying, of course, that no schedule of courses, activities, and other stated requirements will work automatically to produce an efficient teacher. One may even say that a faulty curriculum carried out by scholarly and inspiring instructors would probably produce a better teacher than an ideal program administered by instructors who had no interest in their task and who did their work in a listless and perfunctory way.
>
> The professional program for prospective teachers, then, is something more than its curriculum offerings or requirements. It includes the life of the school itself, which should provide many opportunities for informal contacts between instructors and students, and also the teaching which goes on in the classrooms. Enthusiasm for one's work and devotion to the interests of the learner are qualities of the artist teacher for which there are no substitutes—and these qualities are not taught to the novice, they are rather caught from his instructors if his instructors exemplify them in a striking way.
>
> The ideal preparation of the teacher, then, would include the privilege of working with such instructors.

This takes us a step further. To achieve better teachers for secondary schools demands thoroughly efficient, enthusiastic and inspiring teachers in our colleges, teachers colleges and normal schools. The crusade for well trained teachers goes all the way through our institutions of learning.

The Vital Need

The vital need of the world to-day is the possession of the ability to think and the doing of thinking on the part of every normal individual. Considering the present economic condition of the

[51] "The Ideal Preparation of a Teacher of Secondary Mathematics from the Point of View of an Educationist," The Mathematics Teacher, Vol. XXVI, Number 5, May, 1933.

[52] The Etude Music Magazine, May, 1933, p. 312.

[53] William C. Bagley, op. cit., p. 276.

world, this ideal has not, in the past, been achieved on the part of enough individuals to provide a majority.

Why is a Majority of Logical Thinkers a Vital Need?

Wherever the vote of the people is a vital factor in the government of a nation, the public meeting and the press are, in turn, vital factors in that nation's welfare because the public meeting and the press sway many votes.

In both the public meeting and the press, the speaker or writer needs to be able to demonstrate while his audience needs to understand thoroughly the meaning of a demonstration and to be able to judge whether or not a demonstration is correct. If knowledge of demonstration and reasoning were absent on the part of the speaker or writer, an audience trained in demonstration would sense it, at once, and recognize any errors that his argument might contain. If a speaker or writer coming to logical conclusions through reasoning and demonstration, has an audience untrained in demonstration, that audience will not realize that he is right and his argument will fall on non-understanding ears. In consequence, that audience will be swayed by the wrong as easily as by the right and thus frequently vote for the wrong. Wherever the people have a voice in the country's government, then, it is vital for that country's welfare that its people be able to demonstrate, reason and be able to understand demonstration. Thus training in demonstration is essential to the welfare of the race.

How Can This Training be Obtained?

Mathematics is the one school subject which if properly taught, trains the individual in the power to think and to demonstrate and thus fits him to judge between the right and the wrong, between valid and inferior argument. Mathematics has not had a chance to do this, in the past, because it has not been available to each person capable of receiving it.

In this connection, let us consider one branch of mathematics—demonstrative geometry. I have found from the personal experience of teaching geometry to girls of from eleven to sixteen years of age, that demonstrative geometry can be taught in such a way as to engender, in a short time, such an enthusiasm for demonstration that students beg for originals and continue to do so throughout the course. To do originals, one has to think and the doing of

originals trains in logical thinking and demonstration. An excellent introduction to demonstrative geometry that will achieve this enthusiasm and joy of discovery along with training in demonstration is given in "Demonstrative Geometry for the Ninth Grade." THE MATHEMATICS TEACHER, XXVI, Number III (March, 1933) by W. D. Reeve.

Training in demonstration continues everywhere, throughout the study of mathematics.

Modern aims, if financial conditions allow them to be realized,[54] should extend the *range* of students by whom geometry and mathematics with their training in logical thinking and demonstration can be assimilated. This should help in the achievement of the goal of having the majority of our grown-ups in possession of the ability to think logically, the ability to demonstrate and to understand a demonstration. Certainly, if we are to learn to do these things, we must begin in childhood and constantly practice the accomplishment.

It should be remembered that an essential part of this training is *actual practice*. We must definitely use what is right or what we are convinced is right as the basis of our own demonstrations and demand that others do the same in all those demonstrations that affect the welfare of the world. We must remember, too, that when valid conclusions have been reached, it is only if they are acted upon that they can help the race.

Can the Modern Aims be Achieved?

The ideals of modern education, then, are high. We have seen that if they are to be attained, we require, as instructors, well-trained teachers who know mathematics thoroughly and enjoy teaching it. But these well-trained teachers, however well-trained they may be, cannot perform this task if they are the unemployed and untrained teachers are the employed.

This turns a grave responsibility to the shoulders of those who do the hiring of teachers. It is only by hiring teachers because of their superior qualifications for the particular positions to be filled and by eliminating choice of the unqualified by "pull" that the youth of our day can be given training in the power to think, the

[54] At the present time, some schools and colleges have been closed and the staffs of others seriously cut through lack of funds.

ability to demonstrate and to understand demonstration. The present economic crisis has shown these powers to have been woefully lacking in our generation and essential for the welfare of our citizens of the future.

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Mathematics Teams to Motivate Drill Work in Junior High Schools

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THE JUNIOR HIGH SCHOOL STUDENT is gregarious. He is plastic, he is energetic, he is ambitious, and he is quite sensible. His gregariousness, his desire for group work and competition is, however, his outstanding quality. He enjoys all kinds of team work, and he likes to have a definite goal toward which to work. An appreciation of these truths furnishes the key to well motivated and successful work in mathematics.

All junior high school mathematics teachers realize the need of drill in the fundamentals. The hardest thing to teach any person is something which he thinks he already knows. The hardest thing in which to drill pupils is something in which they think they need no drill. The spirit in which the class attacks the drill will probably determine the results more than the content of the drill. We must create a spirit of attack. If other factors are relatively equal, the spirit in which we do anything will determine the results which will accrue from our efforts.

Last year at Mansfeld Junior High School the teachers were required to give daily drills in the fundamentals. All teachers cooperated as effectively as they could, as all realized the basic necessity for this type of work. As the students had had charts, graphs, and records since they were in the third grade, a need for another type of motivation was felt.

The writer planned to create interest in the drill of fundamentals through a contest, not a contest between individuals, but a contest between mathematics teams which would be selected from the various classes. Since there were no rules for team contests, the writer selected a committee of students from his mathematics classes to make the rules. Although the students were made to feel that they were making the rules and that the rules were theirs, the work was closely supervised. The rules were constructed so that the team with several good students would win and not the team with one outstanding individual. The outstanding individual stu-

dent gets his reward by being elected captain. The rules are the results of two years' contests.

The handicaps mentioned in the last paragraph of the rules are of vital importance. At present they are merely a guess based upon the team's ability and the school's classification of its students. Recently a junior high principal suggested that the handicap be based upon the individual student's ability, and as the student gains in speed and accuracy, his handicap should change. This suggestion is very good as each student will have a chance to determine his team's rating. When the handicap is properly worked out, the mathematics contest will reach more of the students below the median of the class where it is needed most. The older and speedier students are glad to give the others a handicap because of its reflection on their team's ability, while the younger students are glad to get the handicap as it gives them a chance to win. Perhaps the best way for the reader to get an idea of a Mansfeld mathematics contest, is to read the rules.

Mansfeld Mathematics Contest Rules*

(Revised for 1932 Tournament)

Article I

(*The Team*)

Each team shall have a captain chosen by the class.

The team shall be chosen by the captain, one member of the class chosen by the captain to help him, and by the teacher of that class.

Each team shall consist of five members including the captain.

The captain shall have complete charge of the team during the contest. The teacher may give suggestions to the captain at any time.

Article II

(*The Contest*)

The contest shall be between two teams.

Each contest shall consist of three problems of each of the four fundamentals.

The problems in addition shall consist of three digits in the number and nine numbers in the problem. The subtraction problem shall be in hundred millions. The multiplicand shall consist of four digits and the multiplier shall be three digits. The divisor shall have three digits, and the quotient three digits. There shall be no fractions nor decimals. (quotients must be even)

* The rules have been made by the students. Thelma McMillan, an eighth grade student, has been the outstanding leader in their construction. The mathematics teachers are always present when the rules are changed, but they have no vote. The students have always been glad to get suggestions from the teachers.)

Two members shall go up from each team for every kind of problem, making four at the blackboard at one time.

No member of either team shall be permitted to work more than six problems.

The contest shall be judged by three persons, at least one of whom should be a teacher

The teacher in charge of the contest shall read the problems one at a time and declare the score which must be based on the decision of the judges. The teacher in charge must be the teacher of both teams or of neither team.

ARTICLE III

(*The Score*)

When the problem is finished, the contestant must turn and face the judges with his writing hand and chalk above his head. After this gesture the contestant will not be permitted to work nor change the problem.

The first person who finishes with the correct answer will count five points for his team, the second three points, and the third one point. The judges will decide who is first, second, and third.

If every contestant has an incorrect answer, another similar problem will be given for that problem.

If the score is tied at the end of the contest, the teacher in charge will ask each captain which kind of problem his team prefers to work to decide the contest. If the captains want the same kind of problem, one problem will be given to decide the contest. If the captains want different kinds of problems, one of each kind suggested by the captains will be worked. If the contest is still tied after two problems are worked, the teacher in charge will give two more problems, one of each of the other two fundamentals.

The following handicap scores shall be placed on the scores of the first teams before each contest: 8A3—0; 8A 2, 8A 1 and 7A1—5; 8B1—10; 7A 2—25; 7B1—30; 7A3—35; and 7A4—40. Every second team in each class shall add 15 points to its first team's handicap to get its handicap, and every third team shall add 25 points to its first team's handicap to get its handicap. The 7A1 second and third teams shall add an additional 5 points to get their handicap.

The contests between various sections are held after school and require from twenty-five to thirty minutes. The English and history teachers are interested enough to help judge the contests. On Friday the shortened mathematics period is used for team practice, which consists of contests between first, second, or third teams, or "try outs" for various positions on the teams. The students also take advantage of the daily ten to fifteen minute drill in fundamentals as it offers a chance to practice and to show the team captains and the teacher the ability of the individual student.

Busy school men and teachers are interested in results. There are very little objective data at present to show results, but an experiment is now being carried on, using one hundred students in an experimental group and one hundred in a control group to test the value of mathematics teams as a means of motivation in

mathematics. The data from the last tests given by the school system show that the writer's mathematics classes stood near or at the top. No class ranked lower than second when compared to eight or ten other similar classes in the system. Under different teachers, in the English and reading tests given that year, these same students ranked near the middle. These are not good data as too many other factors influenced the results.

There is a great deal of interesting subjective data. The most interesting data are the student reactions. The writer only started the mathematics teams the first year. The next year at the suggestion of the principal and after a talk with the superintendent, the writer taught general science. One would naturally expect the mathematics contests to stop. They would have stopped had it not been for the interest shown by the students. The students whom the writer had taught the year before asked Mrs. Bondurant, their new mathematics teacher, if they could have contests. Since Mrs. Bondurant had heard this request several times, she asked about the rules. After she had looked them over, this teacher liked the idea and with the aid of the student interest and enthusiasm, soon had many mathematics teams functioning. By the middle of the second year all the mathematics sections in Mansfeld had teams organized. The student interest had spread into the sections under two other mathematics teachers where the contests had been requested by the students. The interest grew in the mathematics contest until the question was asked by students and teachers, "Can we have a tournament?"

More teams were organized. Captains were requesting slips to bring their teams into the building before school and at noon so that they could practice. Twenty-two teams, or 110 students, about 40% of all the mathematics students, were soon ready for the first mathematics tournament. The function of the tournament was to motivate the team work, while the function of the team work was to motivate the work on the fundamentals.

The student reaction to the mathematics contest when it conflicted with other school activities is quite interesting. When students were on baseball teams and mathematics teams which had contests that conflicted, it was noted that about fifty per cent went to the mathematics contest. One evening while the city junior high basketball championship was being decided on a court at Mansfeld, a mathematics contest was being held inside with all the

regular team members present. At no time did the writer ever notice a student miss a mathematics contest, in which he was a contestant, to see a ball game. The tournaments will have to be carefully scheduled so that they will not interfere with the activities of other departments.

The teachers' reactions are as interesting as the students'. One teacher states that before she used mathematics teams that the giving of the daily drill in fundamentals was her most difficult task, but now it is the easiest. Teachers are quite sure that the needed change in the spirit of attack on the work in fundamentals has been accomplished. Mathematics teachers are looking forward to the time that the standardized tests are given in the system, feeling sure they will show definite achievement. The teachers seem to enjoy the contests as much as the students and do not mind the necessary after school work. One teacher objected to a control group for the experiment as that group could not have mathematics teams. All of the control groups have asked why they could not have mathematics teams.

One realizes the limitations to this type of work. Without a doubt similar contests can be arranged for other kinds of mathematics work. There will soon be some inter-junior high school mathematics contests in Tucson as the students are requesting them. A larger percentage of the students should be organized into the teams. This can only be done by working out more carefully the handicaps so that every child will try "to make" a team and will have a chance to help that team win.

Mathematics teams have been successful at Mansfeld first, because of the careful supervision, of the interest, and of the after school work given by all the mathematics teachers; second, because of the fine student spirit, student enthusiasm, and the desire for group competition; third, because of the wonderful spirit of cooperation shown by the principal and the other teachers who have helped to judge the contest; and fourth, but not least, because they have motivated the work in drills by showing the student his weak and strong points and have furnished him a definite goal toward which to work.

Teaching an Appreciation of Mathematics: The Need of Reorganization in Geometry*

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This paper aims to do three things: first, to present a modern point of view concerning the nature of mathematics; second, to mention some defects in the typical organization of subject matter in high school plane geometry from this point of view; third, to suggest a reorganization of geometry subject matter which may permit a better appreciation of the nature of mathematics than the typical organization.

Implicitly underlying the argument to be presented is this fundamental assumption concerning mathematical education: *Any mathematics course in school or college should aim to give pupils as large an appreciation of the nature of mathematics as is possible.* It is recognized that this is only one side of the purpose of mathematical education; the other side probably concerns the development of power to think mathematically, not only in mathematics but in other fields of life as well. However, the two aims certainly are not independent, for if we can make improvements in realizing one of them, we probably insure greater success in the other. For this reason, and also in order to keep the discussion within bounds, only the appreciation aim will be considered. It is naturally of prime importance, then, to outline a working philosophy of mathematics in the beginning.

I

To the question, "What is Mathematics?", the traditional or dictionary answer is that *mathematics is merely the science of number and space.* Modern writers in the field of mathematical philosophy do not give such a simple answer. If they give what seems to be a short all-inclusive definition, it is really a summary of various distinctive features of mathematics which have previously been explained at some length. It is the tendency to emphasize not

* Based on a paper read before the Association of Teachers of Mathematics in New England, at Cambridge, Massachusetts, January 21, 1933.

so much the specific subject-matter of mathematics—which goes far beyond the notions of "number" and "space," in the intuitive sense of these terms—but rather the abstract method and structure of the subject.

There seem to be three modern schools of mathematical philosophy of particular interest, namely, the *postulational*, the *logical*, and the *formalist* schools.[1] The simplest, least subtle, and best known of the three views is undoubtedly the postulational view. Furthermore, this view is readily compatible with existing divisions of mathematics, whereas a real understanding of the logical and formalist views requires knowledge of elaborate new logical or symbolic disciplines, far removed from the usual mathematical subjects. Consequently, we choose the postulational view as being best adapted to our purpose of developing a working philosophy of mathematics. But, even so, it will be possible to give only a brief and elementary account of this view. A later footnote will mention the ways in which the other views differ from it.

The keynote of the postulational view is that any branch of mathematics must have a starting point somewhere. Not all of the propositions can be proved and neither can all the technical terms used be defined. Some propositions must be left unproved in the beginning, and some terms must be accepted without definition. The unproved propositions, commonly called postulates or axioms, together with the undefined terms, constitute the starting point of the subject. The postulates are not to be considered as self-evident truths, but rather as preliminary agreements about fundamental properties, which are made for the purpose of getting started in the particular branch of mathematics under consideration. They do not even need to *seem self-evident*, but they *are* required to be *consistent*, that is, no one of them must contradict any other.[2] On the foundation thus taken in the beginning, the subject is built up: theorems are deduced from the postulates with the aid of logical

[1] Outstanding names associated with these three schools are Professors E. V. Huntington and Oswald Veblen with the first, Bertrand Russell and Professor A. N. Whitehead with the second, and David Hilbert of Germany with the third.

[2] It is desirable, though not always essential, that the postulates be *independent* and *categorical* as well as *consistent*. The postulates are independent if no one of them is deducible from any of the others, and a number of consistent postulates are categorical if they form the starting point for essentially only one branch of mathematics. Methods have been devised for testing sets of postulates for consistency, independence, and categoricalness. For a discussion of these methods, together with

reasoning, and all the concepts of the subject are defined in terms of the undefined concepts. It is understood that this description applies only to a branch of mathematics in completed form; actual mathematical research, of course, may proceed by other methods than deductive reasoning from a set of postulates—for instance, by the use of trial and error, or generalization. But according to the postulational view of mathematics, any completed mathematical subject must be capable of being organized in deductive form, that is, in the form of a logically connected framework of postulates and theorems. In other words, a completed branch of mathematics exhibits both the deductive method of reasoning and a deductive structure of thought.

We have said that the postulates are not to be considered self-evident truths; in what sense, then, may the theorems be said to be true? It is obvious that the theorems have not been proved true experimentally, nor have they been proved *absolutely* true by the process of deductive reasoning. The *most* we can say is that the theorems are true in relation to the postulates. Consider, for example, the theorem that *the sum of the angles of a triangle is 180°*. If this theorem were absolutely true it would have to be true no matter what the postulates were. But, it is well known that we can develop good deductive geometries, such as the non-Euclidean geometries, in which the sum of the angles of a triangle does not equal 180°. In the so-called hyperbolic geometry the sum of the angles is less than 180°, while in the elliptic geometry it is greater than 180°. The reason is that the postulates which form the starting points of the non-Euclidean geometries differ from the Euclidean postulates in one important respect: in our ordinary Euclidean geometry we have the parallel postulate stating that through a given point there is just one parallel to a given line; in the hyperbolic geometry, on the other hand, the corresponding postulate states that there are two lines through a point parallel to a given line, and in the elliptic geometry the postulate is that there are no

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many examples of their use, consult J. W. Young, *Fundamental Concepts of Algebra and Geometry*, (Macmillan, 1925). For a more detailed application of these methods in the case of the postulates of algebra, cf. E. V. Huntington, "The Fundamental Propositions of Algebra," in *Monographs on Topics of Modern Mathematics*, (J. W. A. Young, ed., Longmans, Green & Co. 1927). For a modern postulational treatment of geometry (but without reference to consistency, independence and categoricalness) cf. O. Veblen, "The Foundations of Geometry" in the *Monographs*.

lines through a point parallel to a given line. It can be shown that the postulates for either of the non-Euclidean geometries are just as sound logically—i.e., just as consistent, or free from contradiction—as the Euclidean postulates. It follows that the non-Euclidean theorems are logically just as valid as the Euclidean theorems. In particular, the Euclidean theorem that the sum of the angles of a triangle is 180° has no claim to absolute truth. The most we can say is, that *if* we are dealing with a particular kind of geometry, Euclidean geometry, whose fundamental properties are stated in the Euclidean postulates, *then* another property of this geometry is that the sum of the angles of a triangle is 180°. Thus, if some one asks which of these three contradictory systems of geometry is the true system, the answer is: *When considered as deductive structures of thought, the three systems are equally true, each system containing its own set of theorems proved on the basis of its own set of postulates.*

If we pass outside the domain of thought, and ask which system of geometry is true of *physical space*, the answer is harder to give. We find that for practical, everyday purposes, Euclidean geometry is the simplest and most convenient kind, but when we consider the physical space of the astronomical universe, it is quite possible that one of the two non-Euclidean geometries, or some other geometry, may give a better description of the facts. In any case, the truth of any deductive system of geometry, considered in relation to its postulates, is not affected one way or the other by the degree to which that particular system is successful in describing physical space.

One further qualification of the extent to which a mathematical deductive system is true does seem necessary, however. For, in obtaining the theorems from the postulates logical reasoning is used, and the methods of reasoning are simply *assumed* to hold good. Consequently, we ought to say that the theorems are true in relation to the postulates only if we grant the validity of the logical principles used in proving them. This may mean taking more for granted than might be supposed. There is a great deal of dispute nowadays among logicians concerning what logical principles are legitimate, or concerning the sense in which accepted logical principles can be proved to be legitimate.[3]

We may now summarize our working philosophy, based on the

[3] See, for example, E. T. Bell, *The Queen of the Sciences*, (Williams & Wilkins Co., 1931) pp. 24–26, and pp. 135–137.

postulational view of mathematics, as follows: *Any completed branch of mathematics is a deductive structure of thought exhibiting a logical chain of reasoning from the postulates to the theorems. From the mathematical point of view there is no significance in inquiring into the truth or falsity of the postulates so long as they are mutually consistent. The theorems are not absolutely true, but rather are true only in relation to the postulates and logical methods of reasoning used.*[4]

II

Before proceeding any further it would be well to point out that no high school course in any branch of mathematics can hope to give a satisfactory postulational treatment and deductive development of the subject. For, this would demand a degree of analysis and rigor suitable to philosophically inclined students of the late college or graduate level. But, according to our original assumption, we should try to organize our courses so as to give as large an appreciation of the nature of mathematics as is possible. Our question then is: Is the plane geometry course successfully meeting its obligations in this respect? Undoubtedly the course as usually organized has many merits; but here we wish to point out some ways

[4] The logical and formalist views seem to differ from the postulational view largely with respect to locating the starting point of mathematics, and in their treatment of the problem of consistency. Thus, according to the logical view, the rules of logical procedure should themselves be developed logically from a set of undefined terms and unproved propositions of logic, thereby making mathematics a unified science with a single foundation or starting point in logic. The Whitehead-Russell program developed in their treatise, *Principia Mathematica*, is the classic concrete illustration of this view of mathematics. For an introductory exposition of this point of view, see Bertrand Russell, *Introduction to Mathematical Philosophy* (London, G. Allen and Unwin, 1924) especially Chapters 1 and 18.

The work of David Hilbert and the formalist school does not seem to be well understood in this country yet. Some interpreters say rather blithely that, according to Hilbert, "Mathematics is a game played with meaningless marks on paper." Although the writer does not claim to have a real understanding of the theory, it would seem safer to hazard an interpretation something like this: Hilbert advocates going still further back than the logical school goes, so as to make the ultimate basis of all mathematical knowledge dependent upon pre-logical and pre-mathematical principles of a simple intuitive type—having to do, for instance, with the formal manipulation of mere marks. Those interested in obtaining their own interpretations should consult what Hilbert himself has written on the subject. See D. Hilbert und W. Ackermann, *Grundzüge der theoretischen Logik* (Berlin, J. Springer, 1928) and D. Hilbert, *Die Grundlagen der Mathematik* (Leipzig, B. G. Teubner, 1928.)

in which the typical organization seems defective from our present point of view.

In the first place, consider the introduction to the course. Let us assume that the pupils take the course in the tenth grade after a course in junior high school mathematics. In this earlier work they have presumably made some acquaintance with geometrical ideas on an intuitive and experimental basis. But in the introduction to the tenth grade course, pupils are suddenly asked to adopt an entirely different viewpoint—that of the deductive approach to geometry. The textbook perhaps tells them that it is unsafe to rely on the looks of figures, or to establish results by measurement with ruler or protractor. Rather, in order to arrive at absolute *facts* in geometry, they must start with certain postulates and axioms which are so obvious that they are acceptable without proof; and then by the use of sure methods of reasoning they will be able to prove propositions beyond a doubt. If the introduction is of this nature—and from an examination of various textbooks it seems likely that it often is—then it must be considered a serious defect from our point of view. For, instead of emphasizing the *relative* nature of the truth of geometry, the introduction emphasizes *absolute* truth. Instead of treating the postulates as a convenient starting point for the subject, it leads pupils to consider the postulates as inevitable truths. Naturally, this view is more likely than not to be reinforced later on in the course—for instance, when the pupils have occasion to use postulates in the proofs of theorems.

Next, consider the organization of the early theorems of the course. These theorems, together with originals, give the first opportunity for pupils to use the deductive method in proofs, and for them to appreciate a logical chain of reasoning connecting postulates with theorems. Naturally, then, it is desirable to get off to a strong start. But what actually happens? Many of these first theorems must seem almost as obvious to the pupils as the postulates do. The pupils can see from the figure that if two sides and the included angle of one triangle are equal to two sides and the included angle of another, then the triangles are congruent; they can see from the figures that if a triangle is isosceles the base angles are equal, that an exterior angle of a triangle is greater than either opposite interior angle, or that alternate interior angles of parallel lines are equal; and they see these things in spite of all warnings that it is unallowable to rely on the looks of figures. Any inherent

element of motivation for proving theorems of this type seems lacking—that is, the statements of the theorems do not present much challenge to the pupils, and consequently they do not feel an urgent need for a deductive proof. Furthermore, consider the methods which must be resorted to in order to prove some of these obvious looking theorems. Often these methods seem more unnatural, more subtle, more awkward than the methods used in many of the theorems occurring toward the end of the course. The proofs in the congruence group by superposition, and the indirect proofs of some of the parallel line theorems are cases in point. It is small wonder that some pupils get the impression that the deductive method is a hard way of making simple things complicated, or that it is *much ado about nothing!*

Finally, let us ask if the usual organization of the early theorems —for convenience, let us include all of Book I—contributes effectively to an appreciation of the deductive structure of the subject. A typical sequence of theorems may begin with the theorem about vertical angles, proceed with the first two congruence theorems, then the first isosceles triangle theorem followed by the third congruent triangle theorem. After a while, perhaps there is a theorem that the exterior angle of a triangle is greater than either remote interior angle, followed by theorems about parallel lines, and the theorem about the sum of the angles of a triangle. After that may come a theorem about congruent right triangles and the second isosceles triangle theorem. Then usually follow the parallelogram theorems and the inequality theorems. The two standard locus theorems and theorems about concurrent lines may be included at one place or another. Does this long drawn out procession of theorems, interrupted, of course, by numerous exercises, seem well qualified to give pupils a clear idea of a logical chain of theorems leading from postulates to theorem to theorem? When the end of Book I is reached is it likely that pupils appreciate such a fundamental element of structure as the fact that the angle-sum theorem depends directly on the parallel postulate and indirectly on the theorem concerning the exterior angle of a triangle? True, in the course of the development, when a proof for Theorem 22, say, is needed, pupils realize that they can make use of Theorems 4 and 18 and Postulate 7, but they probably think nothing about what might have happened if Postulate 7 had not been adopted in the beginning, or if Theorem 18 had been postponed until later. Also, they

may not realize why Theorem 22 is included at this particular stage of the game; it may be a key theorem in the later development, a significant theorem in itself, or a minor theorem included for the sake of completeness. Whatever it is, it is a theorem in the book and is to be learned along with the other theorems. Of course, whatever sequence of theorems is used must be planned carefully in conformity with the logical demands of the subject, but in the case of the typical sequence, the order prescribed by the subject, and perhaps by tradition, may often seem quite arbitrary and artificial from the point of view of the pupils.

Thus it is quite possible that pupils simply follow the sequence of theorems in the book blindly, realizing in a vague way that the later theorems depend on earlier ones and on postulates, but without getting a satisfactory appreciation of the entire logical structure. They tend to be passive but not active participants in developing this structure.

To summarize, these seem to be some of the defects in the existing organization of subject matter in geometry from the point of view of teaching an appreciation of the nature of mathematics. First, the introductory point of view in the course is likely to imply that the truths of geometry are absolute truths instead of relative truths, and this point of view is probably maintained throughout the whole course. Second, many of the early theorems of the course seem too obvious to inspire much interest on their own account, and often the methods of proof used in these theorems seem too unnatural and awkward to awaken much enthusiasm for the deductive method of reasoning. Third, the sequence of theorems tends to be arranged to meet the logical or traditional requirements of the subject as seen by the author, and not with a view to obtaining the maximum amount of coöperation from the pupils in developing and appreciating the logical structure.

The second objection may be met by the reply that wisely selected originals or practical problems help to stimulate interest in the deductive method. This is undoubtedly true, but there is room for disagreement concerning the extent to which the remedy is effective. Again, the more recent textbooks present the subject by chapters or units rather than by books, and this procedure is probably helpful for teaching an appreciation of logical structure. But often these units simply follow in succession the various groups of theorems in a standard sequence organized by books. The writer

personally believes the defects mentioned to be serious enough to make desirable a fundamental change of point of view, and a fairly radical reorganization of subject matter in the course.

III

A tentative and incomplete plan for a reorganized course will now be outlined, in the hope that it may have possibilities for remedying the defects that have been listed.

The first step involves a change of program prior to the reorganized course. That is, before pupils start the systematic development of the subject, they are to obtain gradually increasing acquaintance with deductive reasoning in geometry. The reason for this is to avoid the sudden change from intuitive to deductive reasoning which must cause a good deal of disturbance to the pupils just starting the present course in plane geometry. Furthermore, it will make it easier to teach an appreciation of the deductive method and structure, and relative truth, of geometry if pupils have some knowledge of deductive reasoning on which to build.

In the reorganized course the introduction occupies a very important place. Its aim is to furnish both motivation for the whole year's work, and also to give a preliminary point of view toward geometry which is broader than what pupils have acquired in their previous intuitive or demonstrative work. Thus the introduction begins by stimulating the pupil's geometric imagination. It tries to get him to conceive of spaces of various dimensions. It attempts to shake the pupil's faith in the unqualified truth of the theorems of Euclidean geometry by acquainting him with some of the outstanding results of the non-Euclidean geometries. This naturally leads to a discussion of how it is possible for geometries like the Euclidean and non-Euclidean, which contradict each other, to exist at the same time. The answer is reached that their starting points are different. It is pointed out that it is necessary to modify only one postulate of the Euclidean set of postulates in order to obtain a non-Euclidean set, and that each system of geometry proceeds to build up its own group of theorems by deductive reasoning from its own set of postulates. By this time the pupils should be ready to proceed with a systematic development of Euclidean geometry, because they can now see some purpose in this kind of development.

In arranging this systematic development, the writer holds the fairly common opinion that it is desirable to assume more proposi-

tions in the beginning than is usual at present. This has three advantages in helping to remedy the defects of the existing organization: first, it strengthens the point of view that any system of geometry is true at most in relation to underlying postulates; second, the deductive reasoning in the early theorems is often much more simple and direct than what is possible in the usual organization; and third, it enables a much more rapid and clearly directed approach to important and striking theorems.

Thus, after a whole chapter devoted to undefined terms and postulates, the first six chapters of the actual development center around the following topics: (1) the sum of the angles of a triangle and of polygons; (2) introduction to similar triangles, including a brief deductive basis for trigonometry and a proof of the Pythagorean Theorem; (3) congruent triangles and a group of theorems and corollaries about special kinds of triangles—isosceles, equilateral, and 45° and 30°–60° right triangles, with corollaries concerning the values of the sine, cosine and tangent of 30°, 45°, 60°; (4) quadrilaterals and area theorems; (5) constructions, and preliminary circle theorems, relating especially to inscription of regular polygons in a circle; (6) further theorems of special interest, including theorems about a line joining the midpoints of two sides of a triangle, and about concurrent lines of a triangle. This completes the subject matter for Part I of the course, which is thus seen to include theorems chosen from all five books of the present organization. In making the choice for this part, effort has been made to pick out those theorems which seem likely to have most inherent interest, which permit simple straightforward proofs, and which can be organized into fairly short logical units. As far as possible, each unit, or chapter, has some definite goal to reach, which can be used as motivation beforehand. Thus, the pupils can coöperate intelligently in developing the logical structure of each unit, and at the end of the chapter it ought to be easy for them to appreciate the logical structure of the whole chapter by itself, in relation to the postulates, and in relation to theorems from preceding chapters.

Part II proceeds with theorems chosen mainly from Books II through V which seem more advanced, or, in some cases, less interesting on their own account. Thus there are chapters dealing with the following topics: tangents to circles, measurement of angles by arcs, locus theorems, further theorems on similar triangles and similar polygons, and the measurement of the circle.

It would be well to return now to the preliminary chapter dealing with the postulates, to see what propositions were assumed there which are usually proved. The most important are the following: If two parallel lines are cut by a transversal, the corresponding angles are equal, and conversely; the three congruent triangle propositions; the proposition stating that two triangles are similar if the three angles of one are equal respectively to the three angles of the other. The postulate about corresponding angles permits, of course, an extremely easy proof for the alternate interior angle theorems, and then the immediate proof of the sum of the angles theorem in the first chapter of Part I. The similar triangle postulate is responsible for the second chapter and the early proof of the Pythagorean Theorem. The congruent triangle postulates are not used until the third chapter.

Of course, these new postulates are not to be forced down the pupils' throats in the beginning. Because of previous work all the postulates ought to seem natural. But, if not, explanation and informal reasoning would be in order, to make them seem a reasonable starting point. Above all, however, the pupils are to realize that the naturalness of the postulates does not make them absolute truths, and that the system of geometry which can be built up from these postulates is not the only possible system, but that it does turn out to be the one which seems most useful for everyday practical purposes.

This completes the rather sketchy discussion of the plan of reorganization. No doubt there are many shortcomings in this plan. It is presented simply as an attempt to build a new geometry course which will enable pupils to acquire a larger appreciation of the deductive method and structure, and relative truth, of mathematics than seems possible through the present course.

"The Mathematics Teacher" wishes its readers a happy and prosperous New Year!

Some Considerations Appertaining to the Content of High School Mathematics*

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RECENT YEARS have witnessed a change in the content of courses in mathematics for the seventh, eighth and ninth grades. There has been a change not only in content but in the point of view in the teaching of the subject. A study of the mathematics courses offered to first-year students in our various colleges reveals two important changes. First, the elements of analytic geometry and of the calculus are introduced earlier, and second, there is much less emphasis on Euclidean solid geometry. Pupils who do not take this subject in high school often miss it in college, for the number of colleges offering a course in Euclidean solid geometry is fast diminishing.

The changes in the mathematics courses in grades VII, VIII and IX, and the reorganization of work in the first year of college lead one to ask why more changes have not taken place in mathematics courses for the tenth, eleventh and twelfth grades. One answer might be that the requirements of the various colleges are responsible. Certainly there can be very little experimentation or change in content of the mathematics work for these grades unless there is close coöperation between colleges, high schools and examination boards.

At the present time, the work in the tenth, eleventh and twelfth grades consists of one year of plane geometry, half a year of solid geometry, one year or half a year of intermediate algebra, and a half year each of advanced algebra and trigonometry. In most schools pupils are allowed to take solid geometry, trigonometry and advanced algebra in any order. For the most part, high schools follow one of two plans in the arrangement of mathematics work:

* An address given at the Spring Meeting of the Connecticut Valley Section of Teachers of Mathematics of New England, Mount Herman, Massachusetts, May 13, 1933.

	First Plan	*Second Plan*
Ninth Grade	Elementary Algebra	Elementary Algebra
Tenth Grade	Plane Geometry	Intermediate Algebra
Eleventh Grade	Intermediate Algebra	Plane Geometry
Twelfth Grade	Solid Geometry and Trigonometry	Solid Geometry and Trigonometry

Objections may be raised to both of these schedules. In the first place, under both arrangements the subjects are taught independent of each other, with the result that the concepts developed in one field cannot be applied to any extent in the other fields. In the second place, both of these plans place a distinct handicap on those students who intend to continue their work in mathematics, for, under the first arrangement, an interval of at least fourteen months elapses between the completion of algebra in high school and the beginning of freshman mathematics work in college, and under the second plan there is an interval of twenty-six months. In the third place, the rigidity of the courses of study prohibits many teachers any play of initiative in the matter of introducing related topics other than those specifically required. And lastly, the course ends without giving the pupil a viewpoint of mathematics as a whole. The goal seems to be either the ambiguous case in trigonometry or some theorem or theorems in solid. Compare this end with courses in English or Latin often found in high school. I have noted in a number of cases a course in English ending with a survey of world literature, or a course in Latin which ostensibly is nothing more than a course in Vergil but which in actual practice is much more, giving the pupils considerable material in history and philosophy.

Of course, a number of pupils do not complete these four years of mathematics, this completion or non-completion depending upon their interest and the requirements of the various colleges. In several schools a review course is offered in the senior year for those pupils who have had a year of plane geometry and two years of algebra. This review course could be one of the most worthwhile courses in mathematics, but it often turns out to be nothing more than a drill for specific examinations. If only this course could be devoted to an excellent review of mathematics as a whole, giving the pupils the place of mathematics in modern society together with its contributions to civilization, the situation would be greatly improved.

In recommending changes in the above course of study, it is

necessary to have clearly in mind definite aims to be obtained. Aims for mathematics instruction have been stated in a number of places; for example, one is referred to in the report of the National Committee on Mathematical Requirements. This bulletin was published under the auspices of the Mathematical Association of America. In stating the courses of study for the various grades, the writer's aims will be indicated or touched upon in general terms.

The Ninth School Year or Orientation Course in Mathematics

1. This year should be required and be the only year that is required.
2. The goal during this year should not be an aimless accumulation of special mathematical facts, but more of an experience in the quantitative aspects of our civilization.

By "2" I mean a little of the history of civilization, the limited mathematical needs of primitive people, and, as civilization grows more complex, the increased need to express quantitative relationships more definitely. This would afford a study of the graph, the formula and the equation. This year should concern itself with units of measure and the advantages of an international system of weights and measures. Further consideration should be given to taxation and its various ramifications. At this point some mention might be made of community problems such as the following: If the breadwinner of a family dies, who takes care of his immediate relatives? Questions that might arise here would be: Was he covered by insurance? If not, how does the community take care of his family? This one problem alone leads to a study of insurance, taxation, and the work of charitable organizations. Some thought should be given to the problem of financial security of an individual. Something should also be included about geometric forms in this year.

The plan suggested implies that the amount of algebra covered is of minor importance as compared with the need for making the pupils conscious of the quantitative aspects of the world in which they live. This ninth grade implies that a certain amount of arithmetic will be studied.

The point of view set forth in the foregoing statement is adequately presented in the writings of Dr. Thomas Jesse Jones, Educational Director of the Phelps-Stokes Fund:

> Strange to say, educators have not adequately and sincerely sought the answer to the most fundamental question of all, namely, What is education? Briefly and emphatically the answer is that education is the preparation of the youth for life, both as individuals and as members of human society. Elemental and obvious as

this answer appears, teachers are not receiving adequate education even in the elementary knowledge of social processes. The essentials of community life are health, economic welfare, interracial adjustment, home and family life, and physical and mental culture. The socialization of education requires that these social essentials shall leaven the teaching of every school subject. History should be a record of the way the people have lived rather than the story of fights and quarrels of political rulers. Mathematics should be a quantitative analysis of all phases of social relationships. Every educational process should be tested by the social objectives necessary to human welfare. The task is not impossible. Even the teachers of the elementary classes can become aware of the elementary social forces. They can use the ordinary school subjects as carriers of knowledge and attitudes that help to prepare pupils for community life. Education for life should be the dominant note of all educational method.

It is well to give another quotation which although it does not embody the same point of view, does emphasize the lack of planning in the preparation of our courses of study. This quotation is taken from "The Aims of Education and Other Essays" by A. N. Whitehead, professor of philosophy at Harvard University:

Quadratic equations are part of algebra, and algebra is the intellectual instrument which has been created for rendering clear the quantitative aspects of the world. There is no getting out of it. Through and through, the world is infected with quantity. To talk sense, is to talk in quantities. It is no use saying that the nation is large—How large? It is no use saying that radium is scarce—How scarce? You cannot evade quantity. You may fly to poetry and to music, and quantity and number will face you in your rhythms and your octaves. Elegant intellects which despise the theory of quantity, are but half developed. They are more to be pitied than blamed. The scraps of gibberish, which in their school-days were taught to them in the name of algebra, deserve some contempt.

The question of the degeneration of algebra into gibberish, both in word and in fact, affords a pathetic instance of the uselessness of reforming educational schedules without a clear conception of the attributes which you wish to evoke in the living minds of the children. A few years ago there was an outcry that school algebra was in need of reform, but there was a general agreement that graphs would put everything right. So all sorts of things were extruded, and graphs were introduced. So far as I can see, with no sort of idea behind them, but just graphs. Now every examination paper has one or two questions on graphs. Personally, I am an enthusiastic adherent of graphs. But I wonder whether as yet we have gained very much. You cannot put life into any schedule of general education unless you succeed in exhibiting its relation to some essential characteristic of all intelligent or emotional perception. It is a hard saying, but it is true; and I do not see how to make it any easier. In making these little formal alterations you are beaten by the very nature of things. You are pitted against too skilful an adversary, who will see to it that the pea is always under the other thimble.

I will not attempt to outline in detail the content for this first or orientation course in mathematics, but it must be a course that

keeps within the realities or near-realities of the pupils to whom it is to be taught if it is to be required of all boys and girls in America.

Care should be taken not to include too many topics during this year. It is the experience of the writer that pupils get more satisfaction in mastering a few topics than in covering ground in a multiplicity of topics. Let us not make our philosophy for this year the learning of less and less about more and more. A great many of the social applications of this course are taken care of in the problem material.

Tenth Grade

This tenth grade course in mathematics should not be a required course.

The best point of view that the writer has heard expressed for this year is found in the speech given by Oswald Veblen of Princeton University at the Rice Institute of Texas, and also expressed by him in addresses given at Atlantic City and in New York. The point of view taken was that one of the phenomena of our time is the rapid advance of analytical methods, and that this should affect the way we look at elementary geometry. Professor Veblen pointed out that elementary geometry should be presented in such a way as to prepare students for other sciences which they have to study later, and in which this very geometry is going to be used. This means that the methods of geometry should not be singular ones peculiar to this subject itself, but should be as far as possible methods which can be used over and over again in other branches of education. He did not advocate in his speech the complete elimination of synthetic methods, but that either method should be used that is most advantageous at the time.

I am in hearty agreement with these recommendations, and I feel that if we will look at geometry for the basic ideas, we can cover during this year the essential work of what is ordinarily found in our present courses in plane and solid geometry, together with an analytical approach to a number of topics. This year should be concerned of course with the big idea of congruence. With this idea of congruence comes the idea of one-to-one correspondence. This idea of one-to-one correspondence should also be brought out between the number pairs and points (x, y) on the plane. In enlarging this idea of congruence we come to similarity, and I believe here we

do not need the long list of theorems that are found in the ordinary textbooks to convey this concept. Before the idea of similarity is taken up, reference can be made to three-dimensional space, and at the time similarity is considered a theorem like the following should be studied:

If a pyramid is cut by a plane parallel to the base,

(a) *The altitude and lateral edges are divided proportionally;* and

(b) *The section is a polygon similar to the base;* and

(c) *The area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

Along with these ideas for *congruence* and *similarity* must be considered the ideas of *parallelism, perpendicularity, angle measurement, area* and *volume.* In connection with area I recommend most strongly the consideration of the parabola and the study of maximum-minimum problems by algebraic methods.

In connection with volumes only volumes of the simplest solids should be considered. This year's work should afford many opportunities for functional thinking, and wherever the opportunity arises, the idea of continuity should be brought out. An excellent opportunity is afforded in the angle measurement theorems. The pupils should be shown that the theorems under this topic can be expressed by means of one theorem, namely:

The angle included between two lines of unlimited length that cut or touch a circumference is measured by half the sum of the intercepted arcs. (Here the word "sum" means algebraic sum.)

I hope that when some new course of study in geometry is offered to the public it will not be just another rearrangement of the theorems in our current textbooks but a fundamental study of what geometry can and should afford the learner. This course of study should not retain anything in the present course just because it has always been studied or because an examining body can form neat questions on the topic.

There have been a number of attempts to develop a course in plane and solid geometry. Most of these attempts have failed and rightly so for two reasons:

1. The purpose of the course was not well defined.
2. The course was not viewed as a whole nor was it viewed in relation to the other years in mathematics.

The main purpose of the work in plane geometry should be the development of logical proof, while in the work in solid geometry, the space concept should receive major consideration.[1]

The Eleventh Grade

This grade should consider the problem of systematizing the algebraic facts previously learned. Emphasis should be placed upon functional thinking. This year should include a study of the exponential and logarithmic functions. Some of the simpler properties of the straight line, the circle, the parabola, the hyperbola and the ellipse should be considered. The work in trigonometry should include the trigonometric functions where the angle is allowed to assume any value. The work should be extended to include at least the Law of Sines, but not the Multiple Angle Formulas or logarithmic work demanding the use of the more complicated formulas.* The use of measuring instruments should be included.

Eleventh Grade Course at The Lincoln School

1. Dependence
   (a) $V = CT$ and $S = 16t^2$—Arithmetic Progressions
   Galileo (1564–1642) Newton (1642–1727) (1)
   (b) Variables, independent and dependent
   (c) From formula to table of values
   (d) From table of values to formula
   (e) Graph of a formula
   (f) Negative numbers
   (g) Graph of linear equations—coordinates
   (h) Graphical solution of linear equations
   (i) Linear equations in the form of $y = mx + b$
   (j) The use of $y = mx + b$ to determine linear relationships (2)
2. Fundamental operations
3. First degree equations and problems
4. Special products and factoring
   (a) Special products
   (b) Factoring

[1] A more detailed outline of this course will be available in the Fall, and may be had upon request. This applies to the Eleventh Grade Course as well.

* I think in this grade it is a mistake to crowd it with the development of a lot of technique. The writer feels that a subject like trigonometry should not be completed in this year. More consideration should be given to the functional idea and the properties of various functions with their occurrence in astronomy and other branches of science. This is infinitely more valuable than developing a technique for finding the remaining parts of a triangle when two sides and the included angle are given, etc.

(c) Graphical solution of quadratic equation
(d) Solution of quadratic equation by factoring
(e) Remainder theorem
(f) Factor theorem
(g) Synthetic division
(h) Factoring x^n-y^n and x^n+y^n
(i) Graph of $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=y$
(j) Graphical solution of $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots a_n=0$

5. Fractions and fractional equations
6. Simultaneous linear equations and linear function
   (a) Algebraic and graphical solution
   (b) Properties of a straight line $y=mx+b_1$
   (c) $m=\dfrac{y_2-y_1}{x_2-x_1}$ (3)
   (d) General definition of tangent
   (e) Derivation of $y-y_1=m(x-x_1)$
   (f) Problems
7. Exponents and radicals
8. Quadratic functions and Equations
   (a) Graphical solution
   (b) Algebraic solution—radical, imaginary numbers
   (c) Elementary properties of the following loci (4)
      (1) Parabola
      (2) Circle
      (3) Ellipse
      (4) Hyperbola
   (d) Extension of the locus work of plane geometry by means of this work in analytics
9. Exponential and Logarithmic Functions
   (1) Exponential functions
   (2) Logarithms
   (3) Properties of the logarithmic function (5)
10. Trigonometry
   (1) Solution of right triangle by natural functions and by logarithms
   (2) Trigonometric function of the general angle
   (3) The Law of Sines
11. Progressions
12. Binomial Theorem and Investments

Notes on the Eleventh Grade Course

(1) Here some attention is given to the state of scientific thought at the time of Galileo and Newton. The development of the theory of arithmetic progressions is considered in connection with the study of falling bodies.

(2) The use of $y=mx+b$ to find the relationship between variables provided the pairs of values arising in a given experiment or

example are the coordinates of points that lie on a straight line. Thus the pairs of values obtained from the comparison of the Fahrenheit and Centigrade thermometers when taken as coordinates of points give rise to points that lie on a straight line. Hence the relationship is in the form $y=mx+b$ or $F=mc+b$. Now if the coordinates of any two points are substituted for F and C, m and b are obtained, and consequently the relationship $F=9/5C+32$. Many other examples can be drawn from science.

(3) The idea of rate of change

(4) Study of some elementary properties by means of analytic geometry.

(5) Consideration of graphs, such as $y=\log_2 x$, $y=\log_3 x$, $y=\log_{10} x$ and the like, and from these graphs the discussion of the following:

1. For any base, $\log_b 1=0$
2. $\text{Log}_b x$ is a real number for positive values of x only.
3. As x approaches 0, $\log_b x$ approaches a large negative number.
4. The logarithm of a positive number greater than unity is positive, while the logarithm of a positive number less than unity is negative.
5. As x assumes different values throughout two intervals of equal length, $\log_b x$ will assume different values and the greatest change in $\log_b x$ will take place in the interval nearer $x=0$.
6. The logarithm of the base itself is unity, or $\log_b b=1$.

The Twelfth Grade

One of the considerations that influenced the author in the following course of study for this grade was the fact that his pupils take the college entrance board examinations. The author chose for the examination Comprehensive CpH. This examination is made up of three parts, namely, solid geometry, trigonometry, and advanced algebra. The writer felt that more freedom was afforded him if his pupils met the two parts, trigonometry and advanced algebra. Meeting this examination also influenced him to this extent, that the experimental work in calculus and analytical geometry for the most part must be completed by Christmas vacation. The time from the beginning of school in January to the close is concerned with the completion of Trigonometry and the study of Advanced Algebra. The work from the opening of school in the fall until Christmas vacation was based for the most part on such texts as: "Elementary Functions and Applications," by Gale and Watkeys (published by Henry Holt and Company) and "An Introduc-

tion to Mathematical Analysis," by Frank L. Griffen (published by Houghton Mifflin Company). Next year these opening weeks will be based upon a new text, "Introductory Mathematics," by Lasley and Browne (published by McGraw-Hill Book Company).

It may be noted here that for anyone contemplating a change in the course of study in the senior year, it would be well to be familiar with the content of the three above-mentioned books. This does not mean that this list is anywhere complete, but it is very suggestive for a new approach to mathematics for this year.

Twelfth Grade Course of Study at The Lincoln School

I. Review of Progressions
II. Mechanics—The consideration of the notion of velocity and acceleration.
III. Functions and their graphs—Variation.
IV. Differentiation of Algebraic Expressions:
    1. Fundamental formulas.
    2. Maxima and Minima.
    3. Applications to mechanics.
    4. Small errors—measurement.
V. Integration of Simple Algebraic Expressions:
    1. Elementary notion—indefinite integral.
    2. Applications.
        a. Areas and volumes
        b. Mechanics—force, work, etc.
VI. Imaginary and Complex Quantities:
    1. Rectilinear and Polar Representation.
    2. De Moivre's Theorem.
VII. Theory of Equations (some use is made of the Calculus).
VIII. Determinants.
IX. Trigonometry completed.
X. Scales of Notation.
XI. Permutations Combinations.
    1. Probability
    2. Statistics.

The writer has taught the above course for five years. With this background in mind, he proposes the following course for consideration provided he was freed from extra-mural examinations or provided there was an extra-mural examination that covered the course outlined below.

Proposed Course

1. The number system of algebra. Complex numbers.

2. Trigonometry as defined by the College Entrance Examination Board, Document number 107.

3. The theory of equations as defined by the College Entrance Examination Board, Document number 107. In addition to the work in the theory of equations the idea of the derivative should be brought in and applied to a number of problems such as maxima and minima.

4. Some work in solid geometry should be included. The purpose of this work is not the development of formal proof but the development of the space concept. The work should not consume more than four or five weeks.

5. The remainder of the year should be devoted to the indefinite integral applied to finding volumes of solids of revolution. Use should be made of the notation of the definite integral. Applications of this should be made to work problems, water pressure, and the like.

The following tentative time allotment is offered: Beginning with the opening of school in the fall, work in trigonometry should be completed by Christmas vacation. The theory of equations and the derivative should be covered in about two months—January and February, for example. Geometry from the Euclidean standpoint could be finished in the month of March and one or two weeks of April. The rest of the year should be devoted to the indefinite integral and its application to volume work and problems from science.

Agreements and Disagreements

For the most part, the author has found that teachers feel that items 1, 2, 3 and 4 should be a part of the senior year's work. A number of people have questioned item 5. This item was questioned for two reasons: (1) The inability to cover this work satisfactorily in the time allowed, or because of the lack of preparation of the teachers. (2) Other persons favor in place of number 5 work in statistics presupposing a knowledge of permutations combinations and probability.

There was another group who favored in place of number 5 a unit in modern geometry. Whether any one of the latter courses is chosen, or item 5 of the proposed course, it will be a more suitable ending for the four years in mathematics in high school than is often found today.

There is a movement on foot which the writer feels tends to defeat the whole purpose of the study of mathematics. Advocates

of this movement recommend the teaching of geometry say twice a week and algebra three times a week over an interval of two years. I do not believe that if algebra and geometry are taught as they should be, this device to keep the two subjects alive throughout an interval of two years is necessary.

History

Quite a little can be done with the history of mathematics in the high school. I find that the pupils are not particularly interested in dates, but they are interested in the spirit of various ages and they are particularly interested in the method of attack used by these earlier peoples in solving their problems. I should like to call your attention to pages 77–87 in the English version of Robert of Chester's Latin translation of the Algebra of Al-Khowarizmi made by Louis Charles Karpinski of the University of Michigan (published by the Macmillan Company, 1915). Many pupils have manifested interest in the Murals depicting principal events in the history of mathematics painted for the Lincoln School by M. Henri Caro-Delvaille. A very good history for high school use is the one by Miss Vera Sanford entitled "A Short History of Mathematics" published by the Houghton Mifflin Company.

Possible Integration of Mathematics and Science

An interesting study could be made in a unit of work which would take one-third of the pupils' time during the eleventh and twelfth grades. This unit of work would consist of mathematics, physics, astronomy, chemistry, and any other phases of mathematical undertakings that society has engaged in. If some such unit were taught by a competent person—a person of broad vision—pupils would more clearly see the place of mathematics and science in the world and the interrelationship of the two.

Psychology vs. Tradition in the Teaching of Algebra

NATHAN LAZAR
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FROM THE VERY FIRST DAY a pupil enters an algebra class it is impressed upon him that there is nothing difficult or mysterious about the subject—that all the operations are patterned after those of arithmetic, and that it is in fact nothing more than a type of arithmetic that employs letters as well as numbers. It is not the purpose of this article to discuss the theoretical relation between these two branches of mathematics or such questions as whether arithmetic derives its underlying principles from algebra or vice versa. There is no doubt however that, from a practical standpoint, the use of arithmetic as a starting point and as a frame of reference will help in the solution of many difficulties that the student of algebra will encounter.

A problem as simple as "By how much does a exceed b?" is well-nigh outside the grasp of the beginner, unless he first substitutes numbers for the letters. He thus becomes conscious of those operations that he unconsciously performs in arithmetic, and, with a little coaxing, will substitute correctly the given letters for the numbers. The same approach is used, as every experienced teacher knows, in teaching the solution of other problems such as "What part of a is b?" "By what number must you multiply m in order to get n?" How much less than a is b?" How much will x apples cost if y apples cost z cents?" It is practically impossible to teach the various operations with algebraic fractions unless constant reference is made to arithmetic. Even the mature mathematician must resort to this analogy whenever he is faced with a new expression or unfamiliar type of problem, or when he attempts to justify to himself or to a pupil a certain mode of approach that he has been using unconsciously.

In view of these known facts it is indeed surprising to find that very few teachers of algebra and still fewer writers of text books utilize the similarity between algebra and arithmetic in the teaching of the four fundamental operations. Far from imitating the

arithmetical procedure, a new method is presented that is actually the reverse of that taught in the elementary schools.

Let us take an example in arithmetic and one in algebra and compare the methods of addition taught in both. To perform the following addition

$$+\begin{matrix}84\\25\end{matrix}$$

the student adds the extreme right hand column first, that is the 4 and the 5, and then proceeds to the addition of the numbers in the left hand column. But when he is confronted with an analogous problem in algebra, such as

$$+\begin{matrix}8a+4b\\2a+5b\end{matrix}$$

he is taught to add the terms in the left hand column first, i.e. the $8a$ and the $2a$ first and then to proceed to the column on the right. Ironically enough most text books affix an example in arithmetical addition as a model for the bewildered student. One text book writer, who evidently was conscious of the difference in approach, took the trouble to rewrite the example in arithmetic in terms of units and tens and proceeded to show that it too could be done by starting at the left. He furthermore gives some more examples in arithmetic and expects the student to imitate the book's procedure and thus become accustomed to the new habit. Clumsy as this method is, it shows nevertheless an insight into the beginner's difficulties and an enviable striving after consistency.

Why should the student of algebra use a new method in adding two expressions? For at least seven years he was drilled to turn immediately to the extreme right hand column whenever the problem calls for addition, subtraction, or multiplication. Why should the teacher attempt to fight a strongly ingrained habit? What exactly is the advantage, theoretical or practical, to be derived by reversing that procedure in the case of algebraic problems? Cannot all the fundamental operations in algebra be performed as expeditiously and as correctly when the arithmetical pattern is followed? Why set up antagonistic optical and muscular habits when nothing is gained thereby? Is it not possible that the difficulty that many students find with the study of algebra is due, in part at least, to an unnecessary tampering with habits firmly established?

Similar remarks may be made in connection with the multiplication of monomial by a binomial such as

$$\begin{array}{r} 3a-4 \\ \times \quad 2a \\ \hline \end{array}$$

Why should not the student be taught to multiply first the $2a$ by the -4 and then the $2a$ by the $3a$, just as he would naturally do in the following analogous example:

$$\begin{array}{r} 34 \\ \times \ 2 \\ \hline \end{array}$$

The usual method of multiplying a binomial by a binomial such as $(2a+5)(a-3)$ is well known.

$$\begin{array}{r} 2a+5 \\ a-3 \\ \hline 2a^2+5a \\ -6a+15 \\ \hline 2a^2- \ a-15 \end{array}$$

Here too most text books give an analogous arithmetical example as a model to be followed, which they honor in principle but violate in practice. Why should not multiplication in algebra follow that of arithmetic?

$$\begin{array}{r} 52 \\ 43 \\ \hline 156 \\ 208 \\ \hline 2236 \end{array}$$

Why not begin in algebra, too, by multiplying first the -3 by the $+5$, then the -3 by the $+2a$, etc. The completed example will appear as follows:

$$\begin{array}{r} 2a+5 \\ a-3 \\ \hline -6a-15 \\ 2a^2+5a \\ \hline 2a^2- \ a \ -15 \end{array}$$

The addition will, of course, start at the right hand column and proceed to the left.*

* The following argument may have occurred to the reader at this point: "The reason for beginning the arithmetical operations of addition, subtraction and multiplication at the right arises out of the need for 'carrying.' Since that need is missing

Let us now analyze in detail the habits formed in the earlier training in arithmetic that are interfered with by the accepted method of multiplying in algebra.

In Arithmetic	*In Algebra*
1. The eyes move to the *right* hand column and the pencil point is poised below it.	1. The eyes have to stop at the *left* hand column, and the pencil point is poised below it.
2. Multiply the *lower right* hand number by the one in the *upper right* and put the answer in the same column below the line.	2. Multiply the *lower left* term by the *upper left* term etc.
3. Multiply diagonally the *lower right* number by the upper *left* and write the answer in the column below the number on the *left*.	3. Multiply diagonally the *lower left* by the upper *right* and write the answer in the column below the number on the *right*.
The fourth and fifth steps follow the analysis of the second and third.	
6. Indent at the *right*.	6. Indent at the *left*.
7. Add the columns, beginning at the *right*.	7. Add the columns beginning on the *left*.

If the above analysis is correct, there are at least seven places where the eye movements and the muscle tendencies of the beginner in algebra function in directions opposite to those he acquired in the study of arithmetic. It surely is not necessary to remind the reader of waste of time and energy entailed in the grafting of a new habit on an old one. But to get an inkling of the difficulty with which the pupil of algebra is faced, let the reader do the following multiplication:

$$\begin{array}{r} 6974 \\ \times 385 \\ \hline \end{array}$$

in performing the same operations in algebra, why not revert to the natural (i.e. the reading) habit of starting at the left?"

The first part of the argument is, of course, true. The second part is however a non sequitur. For, what ever be the historical and practical reasons for the origin of the right-to-left method in arithmetic, it has become an integral part of the child's intellectual apparatus at the age he attempts the study of algebra. It should not be meddled with unless cogent reasons are brought forward for so doing.The argument for the naturalness of the left-to-right method may be met in the following ways:

(1) The right-to-left pattern in arithmetic is as natural (i.e. habitual) to the high school child as the reverse pattern is in reading.

(2) Even to his immature mind, algebra is more closely allied with arithmetic than with reading. It is the habits of the former that algebra should pattern itself after, and not those of the latter.

by beginning on the left hand instead of in the usual way.* Let him observe in particular the initial movements of the eye and the hand that have to be checked, as well as the moments of bewilderment and feeling of helplessness that come over one. Let him multiply all these experiences by ten and he can approximate the predicament of a beginner.

In subtraction, too, tradition has decreed that the process be performed from the left to the right, reversing the procedure followed in arithmetic. There is no need here to repeat the arguments made above. Nor is there any point in making the same analysis of the process of division, since it involves both multiplication and subtraction. It is important, however, to observe that because of its composite nature, the number of habits that would have to be unlearned and reversed when the arithmetical model is not followed in division would be far greater than those found either in multiplication or subtraction alone.

Before coming to a close, it might be well to anticipate some of the objections that will most likely be raised to the suggested innovation. The first one, that of the inveterate standpatter, runs something like this: "Why do you want the change? The methods in use were good enough for past generations, have withstood the test of time and are good enough for us." The writer does not to intend to enter here into a philosophic discussion as to the relative merits of the old and the new. But this much must be said. The unwillingness to try out new methods because those current are old and have come down the ages is as unintelligent and as dangerous as the desire of the immature for constant change. The continuance of established procedures or their abandonment will have to be decided on the basis of utility and not of antiquity. It is well to remember that some of our most cherished institutions like democracy and equality are comparatively young, whereas such scourges of our civilization as prostitution and tyranny are as

* That is, multiply the 3 by the 4, then the 3 by the 7 etc. The indenting is to be done on the left, of course. It may be of interest, to the American reader, to note that the method of multiplication suggested here for experimental purposes is the one in vogue in Great Britain and in continental Europe. See Charles Godfrey and A. W. Siddons "The Teaching of Elementary Mathematics" (Cambridge University Press, London, 1931) pp. 97–8.

For an American exponent of the virtues of this method, see Harry C. Barber "Teaching Junior High School Mathematics" (Houghton Mifflin Co. New York, 1924) pp. 116–121.

old as humanity itself. Let us, however, return to algebra. No doubt many good reasons may be found against the acceptance of the innovation proposed in this paper and for the retention of the current practices. But any reference to the comparative ages of the two methods is like the familiar trick of throwing sand in the eyes of one's opponent. That procedure may end a fist fight, but it will never decide an intellectual battle.

The second objection to the proposed way of performing the fundamental operations in algebra will, most likely, run as follows: "The pupils do not seem to find any unusual difficulty in understanding or in acquiring the accepted method. Why bother changing it?"

It is true that no figures are available to show what per cent of the failures to add, subtract, multiply and divide correctly is due to the interference of an older, dominant habit acquired in the study of arithmetic. But close observation has convinced the writer that it takes even the brighter boys a day or two before they master the new pattern of multiplication. Since at the time that the students are taught the multiplication of a binomial by a binomial they have already learned how to get the product of one monomial by another, it follows that the additional time is used up in the acquisition of a new mode of multiplying.

As an indication of the ease with which pupils learn the proposed method, the writer presents the following fact: out of a class of 32 students who had studied the multiplication of a monomial by a binomial fully 18 did correctly an example that required the multiplication of two binomials *without any instruction whatsoever.*

Perhaps the following may serve as indirect evidence of the difficulties encountered by the average student in the left-to-right method. Teachers are familiar with the pupil who completely forgets the need for multiplying diagonally, and does the following example in the well known manner:

$$\begin{array}{r} 12a+3 \\ 6a-4 \\ \hline 72a^2-12 \end{array}$$

It is highly improbable that that student would make the same mistake in arithmetic and multiply 72 by 39 as follows:

$$\begin{array}{r} 72 \\ \times 39 \\ \hline 2118 \end{array}$$

If he were taught, however, to start his multiplication in algebra on the extreme right hand column, he would most likely proceed in the accustomed manner of multiplying the number in the lower right hand corner by the one in the upper left hand, etc.

"Why don't you devise and perform an experiment to prove your contention instead of making claims for it from a theoretical point of view?" is a third objection likely to be raised. And to that there are three answers possible.

(1) Under the present organization of huge high schools like those prevalent in New York, it is practically impossible to get two groups of students that are matched in intelligence, similarity of home surroundings, etc. Without such a minimum of experimental conditions the results obtained would not be valid.

(2) Every teacher can make a miniature test himself by trying out the suggested method in place of the traditional one. Neither the teacher nor the pupil will be the loser thereby, and both may be gainers. In our zeal for scientifically controlled experiments we often lose sight of the important fact that teaching is primarily an art, and that the collective opinion of the competent is an effective and legitimate way of settling a dispute. The concerted views of a hundred experienced, open minded teachers is not to be treated lightly.

(3) The third and final answer to the above objection is that the burden of proof really lies on those who advocate the continuance of the methods in vogue. For, if our analysis is correct, the teaching of those methods is a flagrant violation of fundamental laws of habit-formation that are universally accepted. There is moreover a stronger argument against their use. For in teaching the traditional methods we fail to inculcate a mathematical attitude that is essential to a complete mastery and understanding of algebra:—that its principles and operations are closely patterned after those of arithmetic!

Be sure to note! The "Official Final Ballot" is on page 63 of this issue.

Brook Taylor

Born at Edmonton, August 18, 1685
Died at London, December 29, 1731

BROOK TAYLOR took an important part in the development of the calculus, but he crowded his chief activity into the period from 1708 to 1719 when he seems to have ceased working in mathematics and devoted himself to writing. He studied at St. John's College, Cambridge; became a fellow of the Royal Society in 1712; and took Newton's part in the Newton-Leibniz controversy. He engaged in a number of challenges with the Bernoullis and was one of the few Englishmen who were able to hold their own against them. He invented the calculus of finite differences, and wrote on the motion of projectiles and the vibration of strings. His *Linear Perspective* (London, 1715) and his *Principles of Linear Perspective* (London, 1719) introduced the subject of vanishing points but much of the work had been anticipated by others.

In Ball's *Mathematical Recreations and Essays*, Brook Taylor is said to have been the originator of the Knight's Move puzzle. The problem is to move a knight to each of the sixty-four squares of a chess board in turn, entering each square once but only once. The problem was solved by De Moivre and at a later date it was treated scientifically by Euler.

In his works on the calculus, Brook Taylor used two different notations for an integral. One was to enclose the quantity to be integrated in a square, a scheme that was clumsy and hard to print. The second method included both derivatives and integrals. The symbols

$$\ddot{x},\ \dot{x},\ x,\ \underset{.}{x},\ \underset{..}{x}$$

stand for successive values of a quantity, each being the derivative of the quantity at its left.

Taylor's *Methodus Incrementorum, directa et inversa* (London, 1715) contains his most important work,—the familiar Taylor's Theorem by which functions may be developed in power series. This theorem was characterized as being fundamental to the differential calculus, by Lagrange. It was first really proved by Cauchy (1789–1857). In discussing this theorem, Klein[1] shows

[1] Felix Klein. *Elementary Mathematics from an Advanced Standpoint.* Translated by E. R. Hedrick and C. A. Noble. New York, 1932. 223–234.

that Taylor derived it from Newton's interpolation formula:

$$f(x) = f(a) + \frac{x-a}{1!}\frac{\Delta f(a)}{\Delta x} + \frac{(x-a)(x-a-\Delta x)}{2!}\frac{\Delta^2 f(a)}{(\Delta x)^2} + \frac{(x-a)(x-a-\Delta x)(x-a-2\Delta x)}{3!}\frac{\Delta^3 f(a)}{(\Delta x)^3} + \cdots.$$

Taylor's scheme was to let Δx become zero and to let n become infinite. Klein calls this "a passage to the limit of unexampled audacity," for while Δx may be neglected, $k\Delta x$ for larger and larger values of k may not be handled in that way. Further, Taylor made no investigations as to the convergency of the series. Klein says, "Taylor really operates here with infinitely small quantities (differentials) in the same unquestioning way as the Leibnizians. It is interesting to reflect that although, as a young man of twenty-nine, he was under the eye of Newton, he departed from the latter's method of limits."[2] Klein also states that at a later point in Taylor's work, he deduced a special form of the series and from the general case

$$f(x) = f(a) + \frac{x-a}{1!}\frac{df(a)}{da} + \frac{(x-a)^2}{2!}\frac{d^2 f(a)}{da^2} + \cdots,$$

by setting $a=0$ he derived the form:

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots,$$

which we recognize as Maclaurin's Theorem. Klein continues the history of this to include Maclaurin's part, "Furthermore, Maclaurin took over this deduction in 1742 in his *Treatise of Fluxions* where he quoted Taylor expressly and made no claim whatever of offering anything new. But the quotation seems to have been disregarded and the author of the book seems to have been looked upon as the discoverer of the theorem. Errors of this sort are common. It was only later that people went back to Taylor and named the general theorem, at least, after him. It is difficult if not impossible, to overcome such deeprooted absurdities. At best, one can only spread the truth in the small circle of those who have historical interests."[3]

VERA SANFORD

[2] Ibid., p. 233. Taylor first announced the discovery of this theorem in 1712.
[3] Ibid., p. 234.

OFFICIAL FINAL BALLOT

For the Election of Officers of the National Council of Teachers of Mathematics at the Cleveland Meeting, February 23, 1934.

For President, 1934–36 (Vote for one)

- ☐ HASSLER, J. O.
  Norman, Oklahoma
- ☐ SANFORD, VERA
  Oneonta, New York

For Second Vice-President, 1934–36 (Vote for one)

- ☐ CONGDON, ALLEN R.
  Lincoln, Nebraska
- ☐ STOKES, C. N.
  Philadelphia, Pa.

For Members of the Board of Directors, 1934–37 (Vote for three)

- ☐ BETZ, WILLIAM
  Rochester, New York
- ☐ CHARLESWORTH, H. W.
  Denver, Colorado
- ☐ CHRISTOFFERSON, H. C.
  Oxford, Ohio
- ☐ KEARNEY, DORA E.
  Cedar Falls, Iowa
- ☐ THIELE, C. LOUIS
  Detroit, Michigan
- ☐ WOOLSEY, EDITH
  Minneapolis, Minn.

Members please mark this ballot at once and mail same to Edwin W. Schreiber, Secretary, 425 E. Calhoun St., Macomb, Illinois. Every ballot must be enclosed in a separate envelope with the name and address of the voter on the envelope. The ballots sent by mail must reach Macomb by February 20. Opportunity for voting will be given to bona fide members attending the annual meeting at the time of registration. The polls will close at 8:00 P.M. Friday, February 23. You must use this official printed ballot or send to the secretary for an official ballot.

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